

Effect of normal current on kinematic vortices

P. Lipavský^{1,2}, Pei-Jen Lin³, Peter Matlock⁴ and A. Elmurodov¹

¹ *Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 12116 Prague 2, Czech Republic*

² *Institute of Physics, Academy of Sciences, Cukrovarnická 10, 16253 Prague 6, Czech Republic*

³ *Department of Physics, Old Dominion University, Norfolk, Virginia, USA and*

⁴ *Research Department, Universal Analytics Inc., Airdrie, AB, Canada*

Within the framework of time-dependent Ginzburg-Landau theory, we discuss an effect of the non-magnetic interaction between the normal current and the supercurrent in the phase-slip regime. The correction due to the current-current interaction is shown to have a transient character so that it contributes only as a system evolves. Numerical analyses for thin layers with no magnetic feedback show that the largest contribution of the current-current interaction appears near sample edges, where kinematic vortices reach maximum velocity.

I. INTRODUCTION

Although time-dependent Ginzburg-Landau (TDGL) theory is justified only for slowly evolving systems, it provides qualitatively correct description of such phenomena as the ultrafast propagation of magnetic flux dendrites^{1,2}, fast kinematic vortices^{3,4} or accelerated vortex motion involving vortex-antivortex annihilation⁵. Of course, in these fast processes the quasiparticles cannot achieve a local equilibrium distribution, meaning that non-equilibrium corrections to the TDGL theory become important. This was demonstrated by Vodolazov and Peeters⁶ who found a large deformation of the gap profile at the phase-slip centre in the case of a slow relaxation of quasiparticles.

Vodolazov and Peeters have assumed isotropic distribution of quasiparticles (valid in the dirty limit) and carefully treated the energy distribution using two coupled kinetic equations for longitudinal and transverse branches. Their approach applies for a finite gap, the time derivative of which acts as a force driving quasiparticles out of equilibrium.

Here we shall discuss a complementary correction (valid in the pure limit) which takes into account a direction-dependent perturbation of the momentum distribution of quasiparticles; such a perturbation appears as the normal current is created. The correction to the TDGL equation is found to be proportional to the scalar product of the normal current and the supercurrent.⁷

indirectly via the magnetic field. The absence of any direct interaction between these two currents in the TDGL theory is not disturbing, because it is in agreement with an intuitive picture based on the two-fluid model of a superconductor: taking the condensate as an independent fluid one expects it not to interact with the underlying crystal including its normal electrons.

The absence of interaction between normal current and supercurrent is also supported by microscopic theories in the dirty limit.⁸⁻¹¹ These approaches, however, cannot be used to discuss the current-current interaction. To obtain practical equations, authors employ the isotropic approximation^{6,9,11} in some cases assuming in addition local equilibrium⁸. The isotropic distribution corresponds to the zero normal current, therefore any effect of the normal current on a formation of superconducting gap is lost in this approximation.

The effect of the normal current on the gap has been derived in [7] from the Thouless criterion¹² adapted to non-equilibrium Green functions. In [13] it was applied to the FIR conductivity of the Abrikosov vortex lattice and shown to explain a decrease of the real part of conductivity below the critical temperature, experimentally observed with ultrafast spectroscopy¹⁴, while the TDGL theory predicts a small increase. Since the microscopic derivation is lengthy and technically demanding, in the appendix we provide a simple derivation of the interaction of the condensate with the normal current using purely phenomenological arguments.

A. Normal current in a superconductor

Superconductors with freely-moving Abrikosov vortices or propagating dendrites have a finite resistivity; an electric field \mathbf{E}' thus develops in them as the current is driven through. This electric field generates a normal current

$$\mathbf{J}_N = \sigma_N \mathbf{E}', \quad (1)$$

which adds to the supercurrent \mathbf{J}_S . In the TDGL theory the normal current and the supercurrent interact only

B. Plan of paper

The paper is organized as follows. In section II we introduce the floating-kernel approximation, which is the TDGL theory extended by the interaction between the normal current and the supercurrent. In section III we show that this correction is of transient nature, being zero in any steady regime. To this end in section III A we perform a gauge transformation to express conveniently the interaction of the normal current and supercurrent in terms of time derivatives of the vector and scalar potentials. Consequently, this correction can be described in

terms of effective magnetic and electric fields as shown in section IIIB. In section IV we apply the theory to the phase-slip regime in a strip made of thin superconducting layers with negligible magnetic feedback. After rescaling so all quantities are dimensionless, in section IV we present results of a numerical simulation to demonstrate how the current-current interaction influences fast kinematic vortices in the phase-slip regime. Section V contains concluding discussion. In the appendix we indicate why the TDGL theory violates the longitudinal f -sum rule and show that addressing this problem with an intuitive two-fluid correction leads to the floating-kernel approximation.

II. FLOATING-KERNEL APPROXIMATION

Here we write down a closed set of equations forming the theory we term the floating-kernel (FK) approximation. This becomes identical to TDGL theory in the coordinate system floating with normal electrons, because the normal current vanishes in this reference frame. When the normal current is accelerated, the condensate experiences inertial forces absent in the laboratory TDGL theory.

A. Order parameter

In the presence of normal current the time-evolution of the order parameter is described by^{7,13}

$$\begin{aligned} & \frac{1}{2m^*} \left(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A} - \frac{m^*}{en}\mathbf{J}_N \right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi \\ &= -\frac{\Gamma}{\sqrt{1+C^2|\psi|^2}} \left(\frac{\partial}{\partial t} + \frac{i}{\hbar}e^*\phi + \frac{C^2}{2}\frac{\partial|\psi|^2}{\partial t} \right) \psi. \end{aligned} \quad (2)$$

The right hand side has been derived by Kramer and Watts-Tobin three decades ago.^{9,15} Terms $\alpha\psi$ and $\beta|\psi|^2\psi$ are a standard part of Ginzburg-Landau theory. The kinetic energy in the left hand side has been obtained only recently.⁷ It applies to pure superconductors, when the Cooperon mass equals twice the electron mass, $m^* = 2m$. In the appendix this normal-current correction is deduced from the longitudinal f -sum rule. The TDGL equation⁹ obtains setting $\mathbf{J}_N = 0$.

Away from the critical line the phase and amplitude relax with different rates. The parameter controlling this difference is $C = 2\tau_{\text{in}}\Delta_0/(\hbar|\psi_0|^2)$, where Δ_0 and ψ_0 are values of the BCS gap and GL function at given temperature in the absence of currents. Since in pure superconductors $\tau_{\text{in}}\Delta_{\text{BCS}} \gg \hbar$, the correction $C^2|\psi|^2$ can be large under realistic conditions. We use this relaxation rate in the numerical example. Close to the critical line this correction vanishes and one can use a simpler theory corresponding to the limit $C \rightarrow 0$ of equation (2).

Our major concern will be the contribution of the normal current, the \mathbf{J}_N -term. In (2) the kinetic energy de-

pends on the difference between the velocity of the condensate

$$\mathbf{v}_S = \frac{1}{m^*} \left(\hbar\nabla\chi - \frac{e^*}{c}\mathbf{A} \right), \quad (3)$$

where $\psi = |\psi|e^{i\chi}$, and the mean velocity of normal electrons

$$\mathbf{v}_N = \frac{1}{en}\mathbf{J}_N. \quad (4)$$

The first term of (2) is thus the kinetic energy which must be yielded by a pair of normal electrons in order to join the condensate, in the reference frame floating with normal electrons. To distinguish the theory based on the velocity difference from the standard TDGL theory, we call equation (2) the floating-kernel approximation.¹³

B. Two-fluid picture of current

The derivative of the kinetic energy with respect to vector potential \mathbf{A} defines the current operator. The correction to the normal current thus also appears in the supercurrent

$$\begin{aligned} \mathbf{J}_S &= \frac{e^*}{m^*} \text{Re } \bar{\psi} \left(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A} - \frac{m^*}{en}\mathbf{J}_N \right) \psi \\ &= e^*n_S(\mathbf{v}_S - \mathbf{v}_N), \end{aligned} \quad (5)$$

with $n_S = |\psi|^2$ being the density of Cooper pairs or the condensate density. That this supercurrent depends on the relative velocity of the condensate with respect to the normal background is desirable. According to Ohm's law (1) all electrons move with the normal velocity \mathbf{v}_N . If a superconducting fraction moves with a different velocity \mathbf{v}_S , we must add the difference.

One may quit the picture of relative motion and rearrange the total current in the spirit of the two-fluid model

$$\begin{aligned} \mathbf{J} &= \mathbf{J}_S + \mathbf{J}_N \\ &= e^*n_S(\mathbf{v}_S - \mathbf{v}_N) + en\mathbf{v}_N \\ &= e^*n_S\mathbf{v}_S + (en - e^*n_S)\mathbf{v}_N \\ &= \frac{e^*}{m^*} \text{Re } \bar{\psi} \left(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A} \right) \psi + \mathbf{J}_N \left(1 - \frac{2|\psi|^2}{n} \right). \end{aligned} \quad (6)$$

In this rearrangement the supercurrent has the condensate velocity \mathbf{v}_S . The correction term has become a part of the normal current, where it reduces the density of electrons to the fraction of normal electrons. We have used that Cooperon charge is twice the electron charge $e^* = 2e$.

The necessity to reduce the normal current to the normal fraction follows from the longitudinal f -sum rule. In the appendix we show that in order to achieve a consistent theory formulated in terms of a free energy, the

reduced normal current must be accompanied by changes in the free energy which lead to the floating-kernel approximation.

C. Scalar and vector potential

The vector and the scalar potentials \mathbf{A} and ϕ yield the electric field

$$\mathbf{E}' = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \quad (7)$$

In some applications one should keep in mind that ϕ is thought of as a local electrochemical potential, and not the electrostatic potential. The vector \mathbf{E}' is thus the driving force per electron rather than the Maxwell electric field. Following the notation of Josephson we write \mathbf{E}' rather than \mathbf{E} as a reminder of this distinction. As is usual in the theory of superconductivity we call \mathbf{E}' the electric field for brevity.

Although the system has non-zero scalar potential, deviations from charge neutrality are so small that one can neglect them using the continuity equation in its stationary form $\nabla \cdot \mathbf{J} = 0$. Substituting for the normal current from (1), one finds the usual condition for the potential

$$\sigma_N \nabla^2 \phi = \nabla \cdot \mathbf{J}_S. \quad (8)$$

We have used $\nabla \cdot \mathbf{A} = 0$ and assumed a homogeneous sample; $\nabla \sigma_N = 0$. To evaluate the vector potential we need the Maxwell equation

$$\nabla^2 \mathbf{A} = -\mu_0 (\mathbf{J}_S + \mathbf{J}_N) \quad (9)$$

which is also in the stationary approximation to be consistent with the continuity equation. The set of equations (1-2), (5), and (7-9) is closed.

III. TRANSIENT NATURE OF THE INTERACTION OF THE NORMAL CURRENT WITH THE SUPERCURRENT

An overlap of the normal current and supercurrent appears at the conversion layer at the junction of the superconductor to a normal lead. Similarly, there is such an overlap at phase-slip centres in superconducting wires or at phase-slip lines in films. In this section we show that the normal current is purely transient and contributes only if the electric and magnetic fields change in time.

A. Effective vector and scalar potentials

The normal current enters the floating-kernel approximation in two ways: in the kinetic energy of the equation (2) and in the supercurrent (3). In both cases \mathbf{J}_N and

\mathbf{A} appear together so that both are accounted for by a vector field

$$\mathbf{A}_{\text{FK}} = \mathbf{A} + \frac{m^* c}{2e^2 n} \mathbf{J}_N. \quad (10)$$

It is advantageous to describe the vector and scalar potentials in a symmetric form. To this end we express the normal current (1) via potentials

$$\mathbf{J}_N = -\sigma_N \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \sigma_N \nabla \phi \quad (11)$$

so that

$$\mathbf{A}_{\text{FK}} = \mathbf{A} - \tau \frac{\partial \mathbf{A}}{\partial t} - c\tau \nabla \phi \quad (12)$$

with the characteristic time

$$\tau = \frac{m^* \sigma_N}{2e^2 n}. \quad (13)$$

By the substitution $\psi = e^{-ie^* \tau \phi / \hbar} \tilde{\psi}$ for a homogeneous sample, $\nabla \tau = 0$, we transform the GL equation (2) as

$$\begin{aligned} & \frac{1}{2m^*} \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A}_{\text{eff}} \right)^2 \tilde{\psi} + \alpha \tilde{\psi} + \beta |\tilde{\psi}|^2 \tilde{\psi} \\ &= -\frac{\Gamma}{\sqrt{1 + \gamma^2 |\tilde{\psi}|^2}} \left(\frac{\partial}{\partial t} + \frac{i}{\hbar} e^* \phi_{\text{eff}} + \frac{\gamma^2}{2} \frac{\partial |\tilde{\psi}|^2}{\partial t} \right) \tilde{\psi}. \end{aligned} \quad (14)$$

with effective potentials

$$\phi_{\text{eff}} \equiv \phi - \tau \frac{\partial \phi}{\partial t}, \quad (15)$$

$$\mathbf{A}_{\text{eff}} \equiv \mathbf{A} - \tau \frac{\partial \mathbf{A}}{\partial t}. \quad (16)$$

The supercurrent (5) then reads

$$\mathbf{J}_S = \frac{e^*}{m^*} \text{Re } \tilde{\psi} \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A}_{\text{eff}} \right) \tilde{\psi} \quad (17)$$

and other equations of the TDGL theory need not be written again since they depend only on the amplitude $|\psi|^2 = |\tilde{\psi}|^2$.

It is now clear that the system behaves as if the normal electrons are driven by potentials ϕ and \mathbf{A} while the superconducting electrons experience effective potentials ϕ_{eff} and \mathbf{A}_{eff} .

B. Effective electric and magnetic fields

The above effective potentials give rise to effective magnetic and electric fields. The transverse component of the normal current acts on the condensate via an effective magnetic field

$$\mathbf{B}_{\text{eff}} = \nabla \times \mathbf{A}_{\text{eff}} = \mathbf{B} - \tau \frac{\partial \mathbf{B}}{\partial t}. \quad (18)$$

The time variation of the normal current acts on the condensate via an effective electric field

$$\mathbf{E}'_{\text{eff}} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}_{\text{eff}} - \nabla \phi_{\text{eff}} = \mathbf{E}' - \tau \frac{\partial \mathbf{E}'}{\partial t}. \quad (19)$$

In both effective fields the correction term vanishes in the stationary limit. The corrections following from the floating-kernel picture might thus become important in transient regimes or in systems driven by oscillating fields. The AC response of the Abrikosov vortex lattice has been discussed in [13]. Here we focus on vortices driven by a steady supercurrent.

IV. FAST VORTICES

When driven by large transport DC , vortices move fast and the condensate is strongly reduced behind the vortex core since it needs a finite time to recover. Similarly, the condensate is stronger on the front side since the condensate needs a finite time to dissolve or move away. In consequence, vortices are so deformed that their interaction becomes anisotropic leading to transition from the triangular Abrikosov lattice to rows perpendicular to the current. These rows are known as phase-slip lines. Along the phase-slip lines vortices move with velocities exceeding typical velocities of normal vortices by one to two orders of magnitude.^{4,5}

In this regime one can expect the floating-kernel corrections to play an important rôle for two reasons. First, in the vicinity of fast vortices time derivatives of the magnetic and electric fields are large so that the effective magnetic field (18) or electric field (19) appreciably differs from the true field. Second, across the phase slip line there is a large normal current since vortices are densely packed there and the condensate is suppressed.

Above a critical current the resistivity of a superconductor changes in dramatic steps corresponding to rearrangement of vortices inside. Presently, this highly non-linear response of the system can be studied only numerically. We thus demonstrate the effect of floating-kernel corrections using numerical simulation.

A. Dimensionless equations for thin films

In films much thinner than the London penetration depth, the current has a negligible feedback effect on the magnetic field. We thus assume a constant homogeneous magnetic field perpendicular to the superconducting film. According to (18) the effective magnetic field coincides with the true magnetic field, $\mathbf{B}_{\text{eff}} = \mathbf{B}$. It is thus convenient to take a stationary vector potential, $\partial \mathbf{A} / \partial t = 0$ in which case from (16) follows that $\mathbf{A}_{\text{eff}} = \mathbf{A}$. We thus focus on the scalar potential and floating corrections to it.

The vector potential is constant and the Maxwell equation (9) is unused. We must solve the TDGL equation

(14) which in dimensionless units reads

$$\begin{aligned} & (\nabla' - i\mathbf{A}')^2 \psi' + (1 - |\psi'|^2) \psi' \\ &= \frac{u}{\sqrt{1 + \gamma^2 |\psi'|^2}} \left(\frac{\partial}{\partial t'} + i\phi'_{\text{eff}} + \frac{\gamma^2}{2} \frac{\partial |\psi'|^2}{\partial t'} \right) \psi', \end{aligned} \quad (20)$$

where the order parameter $\psi' = \tilde{\psi} / \psi_0$ is scaled with the GL value $\psi_0^2 = -\alpha / \beta$. The amplitude of relaxation rate thus reads $\gamma = C\psi_0$. Dimension $x' = x / \xi$ scales with the GL coherence length $\xi^2 = -\hbar^2 / 2m^* \alpha$. The vector potential scales with inverse distance; $\mathbf{A}' = (e^* \xi / c \hbar) \mathbf{A}$. The time $t' = tu / \tau_{\text{GL}}$ is scaled with the GL time $\tau_{\text{GL}} = -\Gamma / \alpha = \pi \hbar / 8k_{\text{B}}(T_c - T)$ and phase relaxation rate u specified below from the scaling of the equation for the scalar potential. The scalar effective potential $\phi'_{\text{eff}} = \phi_{\text{eff}} / \varphi_0$ scales with the inverse time $\varphi_0 = \hbar u / e^* \tau_{\text{GL}}$.

The supercurrent (17) simplifies to

$$\mathbf{J}'_S = \text{Re} [\bar{\psi}' (-i\nabla' - \mathbf{A}') \psi'] \quad (21)$$

with $\mathbf{J}'_S = (m^* \xi / e^* \hbar \psi_0^2) \mathbf{J}_S$. Equation (8) for the scalar potential rescales as

$$\nabla'^2 \phi' = \nabla' \cdot \mathbf{J}'_S, \quad (22)$$

where ϕ' is in the same units as ϕ'_{eff} . We have used $u = \pi \hbar / 4k_{\text{B}} T_c \tau$ with τ from Eq. (13) to make Eq. (22) free of numerical factors. The temperature dependence of τ_{GL} cancels with the condensate density $\psi_0^2 \approx n(T - T_c) / T_c$. The effective potential is

$$\phi'_{\text{eff}} = \phi' - 2 \left(1 - \frac{T}{T_c} \right) \frac{\partial}{\partial t'} \phi'. \quad (23)$$

In the numerical study we use values $u = 6$, $\gamma = 10$ and $T = 0.75 T_c$. The sample has size $20\xi \times 40\xi$ and contacts of length 4ξ are centered at shorter sides. Our numerical code is a modified version of that used by Berdiyrov, Milošević and Peeters. Principally, the modification has consisted of the addition of the normal-current term. The reader interested in details related to boundary conditions can find them in their paper [4] or in a closely related [16].

B. Numerical results

Figures 1 and 2 show the order parameter in a superconducting film in a weak magnetic field $H = 0.1 H_{c2}$ driven by a current from normal contacts. Near the center of the strip the current is strong enough to enforce a rearrangement of vortices into closely packed lines, while near the end points the vortices remain isolated. The strip thus sports both kinetic vortices in lines and isolated Abrikosov vortices.

The left-hand panels show dynamics evaluated within the TDGL theory, the central panels show the dynamics obtained from the floating-kernel approximation. The

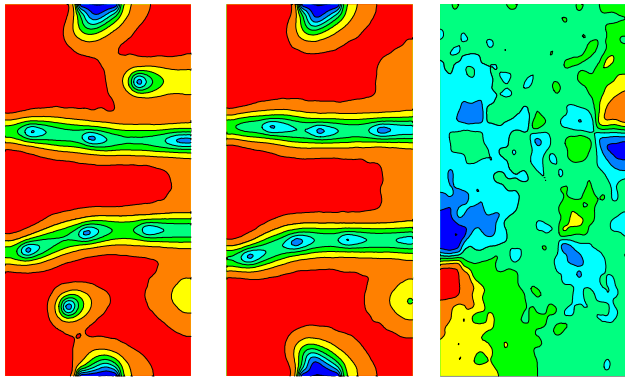


FIG. 1. Effect of the floating-kernel correction on the condensate density in a 2-D sample: Low density is indicated by blue and maximum by red. The left-hand panel shows the TDGL theory while the central panel shows the floating-kernel approximation. The sample is in a weak magnetic field which breaks left-right symmetry. The DC is driven into it from normal-metal contacts creating visible conversion layers at horizontal edges. Near the center of the sample there are two phase-slip lines with kinetic vortices, closer to the contacts one can find isolated Abrikosov vortices. The right-hand panel presents the floating-kernel correction $\tau \partial \phi / \partial t$. It is small in magnitude being a few percent of the scalar potential ϕ . There is no visible contribution due to conversion layers, as this would correspond to the transient nature of the floating-kernel correction. The largest contributions (strong red and blue) appear near the ends of the phase-slip lines, where kinetic vortices achieve the highest velocity.^{4,5}

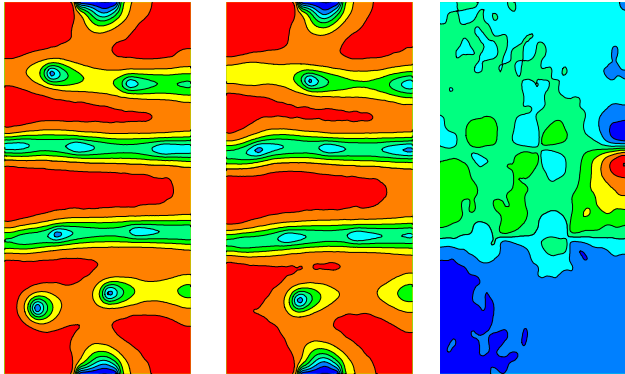


FIG. 2. As in figure 1 but for a current increased by 15%. Again, there are no traces of the conversion layers in the floating-kernel correction. The only appreciable contribution results from the kinetic vortex just at the sample edge. In the upper phase-slip line one can see a four-leafed clover showing the quadrupole character of the floating-kernel correction $\tau \partial \phi / \partial t$ of a single kinematic vortex. Similar features can also be identified for other kinematic vortices inside the sample.

overall behaviour in both approximations is very similar. The floating-kernel correction to the scalar potential leads to slightly different motion of vortices as one can see from their positions after the same time of evolution has elapsed from initial conditions of zero current and zero magnetic field.

The right-hand panel in figures 1 and 2 show the floating-kernel correction to the scalar potential. It is worthy of note that there are no visible contributions near contacts, where the normal current is converted into the supercurrent and these two currents have a strong overlap giving seemingly good conditions for their interaction. This demonstrates the transient nature of floating-kernel corrections, as argued in section III.

It should be noted that for the above regime the true scalar potential has dominant contributions from conversion regions and barely-visible features from phase-slip lines. The amplitude of the floating-kernel correction is only few percent on this scale.

There are no visible contributions to the effective potential from isolated Abrikosov vortices. This is due to their low velocities⁵ which renders their contribution insignificant on the present scale.

Near sample edges the floating-kernel correction to the scalar potential is dominated by kinematic vortices. This corresponds to the acceleration of kinematic vortices to velocities by one order of magnitude larger near edges than in the interior of the sample.^{4,5}

In spite of a weak signal from vortices deep inside the sample, one can see in the right-hand panel of figure 2 that each vortex brings a quadrupole floating-kernel correction to the potential. This can be understood from the Bardeen-Stephen picture.¹⁷ They have shown that a vortex moving in the horizontal direction creates an electric field which drives the normal current through its core in the vertical direction. The corresponding scalar potential is thus a dipole with vertical orientation. The time derivative of this potential due its horizontal translation has quadrupole symmetry.

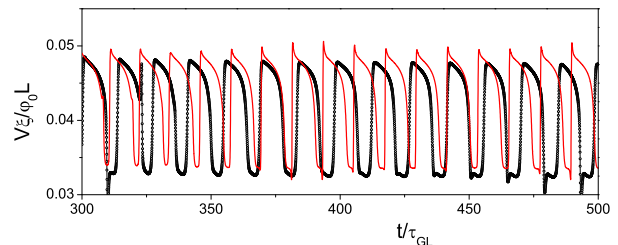


FIG. 3. Voltage across the sample as a function of time: The TDGL theory (black symbols) and floating-kernel approximation (red line) yield similar mean voltage, $V_{\text{TDGL}}\xi/\varphi_0 L = 0.0416$ and $V_{\text{FK}}\xi/\varphi_0 L = 0.0440$. In the floating-kernel approximation the voltage oscillates faster than in the TDGL case. The voltage is a difference of potentials at centres of contacts.

Figure 3 shows the voltage across a sample as a function of time. One can see that the mean voltages obtained by time averaging are rather similar; the floating-kernel gives by 6% higher value than the TDGL theory. As one can expect from the transient nature of the floating-kernel correction, a more pronounced difference appears in the oscillation corresponding to passage of individual kinematic vortices. The floating-kernel approximation as

compared to the TDGL theory yields an increase in frequency of 17%.

The time dependence in figure 3 was simulated under conditions corresponding to figure 1. In figure 1 one finds that the number of kinematic vortices in the TDGL approximation is six while in the floating-kernel approximation there are seven. Since $7/6 - 1 \approx 17\%$ we conclude that vortices move with similar velocities and the different frequency of oscillations follows from higher density of kinematic vortices.

One can understand why vortex velocities are equal for both approximations in the case of an isolated Abrikosov vortex in the following way. The vortex velocity \mathbf{v}_V is given by the driving current density $\bar{\mathbf{J}}_S$ and the friction coefficient η as a balance of the Lorentz and friction force $\bar{\mathbf{J}}_S \times \hat{\mathbf{z}} = \Phi_0 \eta \mathbf{v}_V$, where $\hat{\mathbf{z}}$ is a direction vector either of the true magnetic field \mathbf{B} or of the effective field \mathbf{B}_{eff} . As long as both fields are parallel, the vortex moves in both approximations with the same velocity.

Although the velocity of kinematic vortices does not obey a simple law of balance of forces, applicable to isolated vortices, the numerical result suggests that their velocities are also only negligibly influenced by the floating-kernel corrections. During the time evolution of the simulation, vortex core positions were traced and their velocities calculated. It was found that isolated vortices typically move about ten times slower than kinematic vortices. Since velocities of kinematic vortices change strongly with vortex positions^{4,5} it was not possible to compare the effect on velocities of the two approximations with high accuracy.

V. CONCLUSIONS

Within the floating-kernel approximation we have discussed the effect of the normal current on the motion of the condensate. It was shown that this effect is of transient nature and contributes only in time dependent systems with moving vortices. The most pronounced effects were found for kinematic vortices, which move much faster than isolated Abrikosov vortices. Particularly strong corrections were found near sample edges, where kinematic vortices are accelerated, increasing their speed nearly by an order of magnitude.

ACKNOWLEDGMENTS

Authors are grateful to Tom Lemberger who drew our attention to the longitudinal f -sum rule, and to Golibjon Berdiyev for valuable comments on the manuscript. This work was supported by research plans MSM 0021620834 and No. AVOZ10100521, by grant GAČR P204/10/0687 by DAAD and by Taiwan-Czech PPP project 99-2911-I-216-001.

APPENDIX: FREQUENCY SUM RULE

The conductivity is required to satisfy the frequency sum rule^{18,19}

$$\frac{2}{\pi} \int_0^\infty d\omega \text{Re} \sigma(\omega, \mathbf{k}) = \frac{ne^2}{m}, \quad (24)$$

where e , m and n are charge, mass and density of electrons. To satisfy this sum rule, a modification of the TDGL theory in the spirit of the two-fluid theory is necessary. Here we show that this two-fluid correction implies the floating-kernel approximation discussed in section II.

1. Sum rule violation in the TDGL theory

First we show that the standard TDGL theory leads to a conductivity which violates the sum rule (24). In the standard TDGL theory one writes the total current as a sum of the supercurrent and the normal current,

$$\mathbf{J}_{\text{TDGL}} = \mathbf{J}_{\text{GL}} + \mathbf{J}_N. \quad (25)$$

Neglecting the Hall effect, both currents are parallel to the electric field and the conductivity is a scalar given by the ratio, $\sigma = \mathbf{J}/\mathbf{E}'$. It thus has two corresponding parts

$$\sigma_{\text{TDGL}} = \sigma_{\text{GL}} + \sigma_N. \quad (26)$$

The sum rule (24) is satisfied in the normal state

$$\frac{2}{\pi} \int_0^\infty d\omega \text{Re} \sigma_N(\omega, \mathbf{k}) = \frac{ne^2}{m}. \quad (27)$$

The superconducting component of mean Cooperon density \bar{n}_S , mass m^* and charge e^* has an analogous sum over frequencies

$$\frac{2}{\pi} \int_0^\infty d\omega \text{Re} \sigma_{\text{GL}}(\omega, \mathbf{k}) = \frac{\bar{n}_S e^{*2}}{m^*}. \quad (28)$$

The total sum rule (24) for σ_{TDGL} is thus violated.

2. Two-fluid correction

Assuming that formation of the condensate depletes the density of normal electrons, $n_N = n - 2\bar{n}_S$, the normal conductivity ought to be correspondingly lowered,

$$\sigma = \sigma_{\text{GL}} + \left(1 - \frac{2\bar{n}_S}{n}\right) \sigma_N. \quad (29)$$

The sum over frequencies in the left hand side of (24) is then

$$\frac{2}{\pi} \int_0^\infty d\omega \text{Re} \sigma(\omega, \mathbf{k}) = \frac{\bar{n}_S e^{*2}}{m^*} + \left(1 - \frac{2\bar{n}_S}{n}\right) \frac{ne^2}{m}. \quad (30)$$

A sum rule similar to (30) was discussed in greater detail for the Meissner state, where a part of the weight due to superconducting electrons is covered by a singular term in the form of a Dirac δ function.^{19–21} Avoiding the δ function one arrives at the sum rule¹⁹

$$\frac{2}{\pi} \int_{+0}^{\infty} d\omega \operatorname{Re} \sigma(\omega, \mathbf{k}) = \left(1 - \frac{2\bar{n}_S}{n}\right) \frac{ne^2}{m}. \quad (31)$$

Here we assume similar structure for the mixed state. The only difference is that in the presence of vortices the conductivity σ_{GL} of superconducting electrons is finite even at zero frequency. Its frequency dependence is not a δ function but has a Drude form.

In the pure limit, $m^* = 2m$, the sum rule (24) is satisfied. As one can see, the sum rule (24) is violated in the dirty limit when $m^* \neq 2m$. This is in consequence of limitations of the theory used to derive the floating-kernel approximation. The derivation of [7] is based on the Kadanoff-Baym ansatz with the spectral function approximated by the δ function, therefore the renormalization of the Cooper-pair mass due to finite mean free path is not included in this approach. Briefly, the floating-kernel approximation is justified only in the pure limit.

3. Interaction of normal current with condensate

We note that the floating-kernel approximation discussed in section II leads to the conductivity (29), with current (6). Here we approach the problem in the opposite way: starting from the conductivity (29) we arrive at the floating-kernel approximation.

Let us require that the set of TDGL equations follows from the effective free energy²² \mathcal{F} through

$$\Gamma \left(\frac{\partial}{\partial t} - ie^* \phi \right) \psi = - \frac{\delta \mathcal{F}}{\delta \bar{\psi}}, \quad (32)$$

$$\mathbf{J} = - \frac{\delta \mathcal{F}}{\delta \mathbf{A}}. \quad (33)$$

Since our focus is on the spatial gradients, we neglect the non-linear relaxation of Kramer and Watts-Tobin. Indeed, the relaxation in the left hand side of (32) results from (2), sending $C \rightarrow 0$.

In equations (32–33), the GL function ψ is normalized to the Cooperon density as $n_S = |\psi|^2$ and the sum rule uses the value averaged over space, $\bar{n}_S = \langle |\psi|^2 \rangle$. We assume that the free energy has the superconducting and the normal parts, $\mathcal{F} = \mathcal{F}_N + \mathcal{F}_S$, where the normal part is the same as in the normal state, therefore

$$\mathbf{J}_N = - \frac{\delta \mathcal{F}_N}{\delta \mathbf{A}}. \quad (34)$$

The GL function thus enters the superconducting part

only;

$$\frac{\delta \mathcal{F}_N}{\delta \bar{\psi}} = 0. \quad (35)$$

In the TDGL theory the supercurrent reads

$$\mathbf{J}_{\text{GL}} = \frac{e^*}{m^*} \operatorname{Re} \bar{\psi} \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A} \right) \psi. \quad (36)$$

Since the vector potential \mathbf{A} appears exclusively via the covariant gradient in the bracket, the current implies that in the TDGL free energy the kinetic energy takes the familiar form $(1/2m^*) |(-i\hbar \nabla - (e^*/c)\mathbf{A})\psi|^2$.

We have seen that the current (36) with the normal current added violates the frequency sum rule. Now we derive the kinetic energy assuming that current includes the two-fluid correction. According to the two-fluid conductivity (29), the total current reads

$$\begin{aligned} \mathbf{J} &= \sigma_{\text{GL}} \mathbf{E}' + \left(1 - \frac{2n_S}{n}\right) \sigma_N \mathbf{E}' \\ &= \mathbf{J}_{\text{GL}} - \frac{2n_S}{n} \mathbf{J}_N + \mathbf{J}_N \\ &= \frac{e^*}{m^*} \operatorname{Re} \bar{\psi} \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A} - \frac{m^*}{en} \mathbf{J}_N \right) \psi + \mathbf{J}_N. \end{aligned} \quad (37)$$

We have included the correction term in the supercurrent, because it is proportional to the condensate density. According to (33) and (35) it thus cannot result from the variation of the normal free energy \mathcal{F}_N .

From equations (33), (34) and (37) one finds

$$\frac{\delta \mathcal{F}_S}{\delta \mathbf{A}} = - \frac{e^*}{m^*} \operatorname{Re} \bar{\psi} \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A} - \frac{m^*}{en} \mathbf{J}_N \right) \psi. \quad (38)$$

Integrating relation (38) over the vector potential one finds the superconducting free energy of form

$$\begin{aligned} \mathcal{F}_S &= \frac{1}{2m^*} \left| \left(-i\hbar \nabla - \frac{e^*}{c} \mathbf{A} - \frac{m^*}{en} \mathbf{J}_N \right) \psi \right|^2 \\ &\quad + \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4. \end{aligned} \quad (39)$$

Of course, the integration provides only the kinetic energy which has to be rearranged with integration by parts into the square of covariant gradients. The terms independent of \mathbf{A} represent initial conditions for the integral and are taken from the standard GL theory.

With free energy (39), equation (32) is identical to the floating-kernel approximation (2) in the $C \rightarrow 0$ limit. It should be noted that derivation of equation (2) from microscopic theory was also carried to terms linear in \mathbf{J}_N , therefore additional terms quadratic in the normal current might appear.

To summarize this appendix, we have shown that the TDGL theory violates the longitudinal f -sum rule. To restore the sum rule one must reduce the normal current which corresponds to the interaction term between the normal current and the condensate. In this way one recovers the floating kernel-approximation from phenomenological arguments.

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